ON THE GENERALIZED LAMBERT FUNCTION

ALEXANDER KREININ*, ANDREY MARCHENKO, AND VLADIMIR V. VINOGRADOV

Abstract. We consider a particular generalized Lambert function, $y(x)$, defined by the implicit equation $y^\beta = 1 - e^{-xy}$, with $x > 0$ and $\beta > 1$. Solutions to this equation can be found in terms of a certain continued exponential defined by formula (3.10). Asymptotic and structural properties of a non-trivial solution, $y_\beta(x)$, and its connection to the extinction probability of related branching processes are discussed. We demonstrate that this function constitutes a cumulative distribution function of a previously unknown non-negative absolutely continuous random variable.

1. Introduction

The Lambert W function has a long history going back to 18^{th} century. Nevertheless, it was rigorously introduced in 1996 only and studied relatively recently. This function is closely related to the logarithmic function and arises in many models of the different areas of sciences, including a number of problems in ecology and evolution theory. The main reason why the Lambert W function became so popular is because this function allows us to solve explicitly important models for which this is not possible in terms of elementary functions.

The present paper is partly devoted to the interplays of the classical analysis, in particular, transcendental functions, probability theory as well as certain areas of computational mathematics, namely iteration schemes, Monte Carlo simulation and evolution of dynamical systems.

The classical Lambert W function is defined as (the multi-valued) inverse function to

$$y = x \cdot e^x,$$

(see [7]) and therefore, it satisfies the following implicit equation:

$$W(x) \cdot e^{W(x)} = x.$$

This function was introduced by Lambert [16] in the second half of 18^{th} century and became a subject of Leonard Euler's studies [9] on the transcendental equations. Since then thousands of papers dedicated to the Lambert function and its generalizations were published.

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* Corresponding author.

The function W, has many applications in various areas including mathematical and theoretical physics [7], [8], [17], [19]. This function is deeply connected to many explicitly solvable models and problems ranging from population growth rates to fertilization kinetics and disease dynamics [5], [17], [18], [19], [22], [23].

The function W provides an exact solution to the equation

$$e^{-cx} = a \cdot (x - b)$$

(see [7]). The function W_p, the principal real branch of the Lambert W function, has the Taylor series expansion

$$W_p(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n,$$

with the radius of convergence, $\rho = e^{-1}$. It also admits the following *continued exponential* representation [7]

$$W(x) = x \cdot e^{-xe^{-xe^{-x\cdots}}}. \tag{1.1}$$

Generalizations of the Lambert W function were also relatively recently considered. In particular, in quantum mechanics research on Bose-Fermi statistics the equation

$$e^{-cx} = a \cdot \frac{P_n(x)}{Q_m(x)},$$

where P_n and Q_m are particular polynomials of degrees n and m respectively, was analyzed [20]. In [5], [6] applications of the generalized Lambert functions are recently seen in the area of atomic, molecular, and optical physics.

In the theory of branching processes, a generalization of the Lambert equation describes the extinction probability of an explosive Galton-Watson branching process.

Probabilistic flavour of the solution to the generalized Lambert equation (see Section 2) naturally leads to an interesting family of random variables that we denote ξ_β in what follows. The moments of ξ_β can be computed in closed form. The study of the analytical properties of the moments lead to an elegant area of mathematics called Tornheim series (see [25] as well as [1], [2], [3], [4], [15]).

The structure of the present paper is as follows. In Section 2 we discuss connection between the generalized Lambert function and the extinction probability of the Galton-Watson process. In Section 3 the main analytical properties of the generalized Lambert functionare discussed.

In Section 4, we describe the asymptotic expansions and two-sided bounds for the generalized Lambert function. In Section 5, the numerical examples of the computation of the generalized Lambert function are discussed. A probabilistic interpretation of the generalized Lambert function and characteristics of the corresponding random variables are discussed in Section 6. Some technical statements are deferred to the Appendix.

To conclude the Introduction, we summarize some relevant notation and terminology. Hereinafter, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^1_+$, $\mathbb{Z}$, $\mathbb{Z}_+$, stand for the sets of all complex numbers, all real numbers, all positive reals, all integers, and all non-negative integers, respectively. In what follows, we use the notation $F_\eta(x)$ for the cumulative distribution

function (or cdf) of the random variable (r.v.) η. The symbol "$\bar{F}_\eta(x)$" stands for the survival function, $\bar{F}_\eta(x) := 1 - F_\eta(x)$. The notation "$\mathbb{E}[\eta^m]$" is used for the mth moment of the random variable η, $m \in \mathbb{Z}$. The natural logarithm of x is denoted by $\log x$. The Gamma function is traditionally denoted by $\Gamma(x)$, $x \in \mathbb{R}$, $x \geqslant 0$. The Riemann zeta function is denoted by $\zeta(z)$, $z \in \mathbb{C}$. If z is real and $z > 1$, $\zeta(z) := \sum\limits_{n=1}^{\infty} \frac{1}{n^z}$.

2. Extinction probability of a Galton-Watson process

Extinction probability of a Galton-Watson process satisfies the equation

$$\pi_\nu(z) = z, \quad 0 \leqslant z \leqslant 1, \tag{2.1}$$

where $\pi_\nu(z)$ is the probability generating function of the number of offsprings, ν, in the branching process (see [10], [12], [13]). If the random variable (r.v.) ν has a Poisson distribution with parameter $\lambda > 0$, then Equation (2.1) takes the form

$$e^{\lambda(z-1)} = z,$$

and can be transformed to

$$xe^x = -\lambda \cdot e^{-\lambda}.$$

The connection with the classical Lambert W function and the Lambert equation becomes clear in this case.

If the distribution of the r.v. ν belongs to the class of *discrete stable distributions* the probabilities $\mathbb{P}(\nu = k)$, are defined by the equation

$$\mathbb{P}(\nu = k) = \frac{1}{k!} \cdot {}_1\Psi_1(-\alpha, k; -\alpha, 0; \lambda),$$

where ${}_1\Psi_1$ is a particular case of the Wright function, such that given $\delta \in \mathbb{C}$, $\rho \in (-1, 0) \cup (0, \infty)$, argument $z \in \mathbb{C}$ with $|z| < \infty$ and *real* $k \geqslant 0$,

$${}_1\Psi_1(\rho, k; \rho, \delta; z) := \sum_{n=0}^{\infty} \frac{\Gamma(\rho n + k)}{\Gamma(\rho n + \delta)} \cdot \frac{z^n}{n!}. \tag{2.2}$$

A discrete stable r.v., ν , with parameters $\lambda > 0$ and $\alpha \in (0, 1]$ is characterized by its p.g.f., $\mathcal{P}_\alpha(u) := \mathbb{E}[u^\nu]$

$$\mathcal{P}_\alpha(u) = \exp\{-\lambda \cdot (1-u)^\alpha\}. \tag{2.3}$$

The well-known case $\alpha = 1$ corresponds to the subclass of *Poisson* distributions which are naturally incorporated to the family of discrete stable laws.

The probability of ultimate extinction for a supercritical Galton-Watson process with discrete stable branching mechanism that starts from one particle appears to be intimately connected to a generalized Lambert function. Indeed, Equation (2.1) can be written as

$$e^{-\lambda(1-u)^\alpha} = u.$$

Denote $x = (1-u)^\alpha$. Then $1 - u = x^\beta$ and the latter equation can be written as follows:

$$e^{-\lambda x} = 1 - x^\beta, \quad x \in (0, 1], \quad \lambda > 0, \tag{2.4}$$

where $\beta = \alpha^{-1}$.

The following rather elementary assertion stipulates the existence and uniqueness of the solution to (2.4).

Proposition 2.1. *Let* $\lambda > 0$ *and* $\beta > 1$. *Then Equation* (2.4) *has the unique solution in the open interval* $(0, 1)$.

Proof. Indeed, consider the function

$$\mathcal{F}(x) := e^{-\lambda x} - 1 + x^{\beta}, \quad \beta > 1. \tag{2.5}$$

We have

$$\mathcal{F}(0) = 0, \quad \mathcal{F}(1) > 0, \quad \text{and the derivative } \mathcal{F}'(0) < 0.$$

The second derivative $\mathcal{F}''(x) > 0$. Therefore, the function $\mathcal{F}$ is convex in the interval $[0, 1]$ and Equation (2.4) has a unique solution, x_0, satisfying the inequalities $0 < x_0 < 1$. □

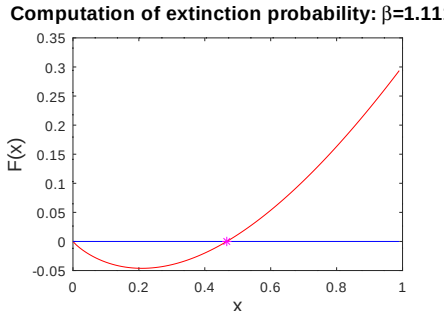

FIGURE 1. The function $\mathcal{F}(x)$.

Here, the root of the equation $\mathcal{F}(x) = 0$ is shown by the star in Figure 1. For $\beta = 1.11$ and $\lambda = 1.2$, the root $x_0 \approx 0.46671$. The root of Equation (2.4) is found by the iteration process described by Theorem 3.7 in the next Section. The iteration process is stopped once the iterations, x_n, satisfy the inequality

$$|\, x_n - x_{n+1} \,| \leqslant \varepsilon,$$

where $\varepsilon = 10^{-6}$. For the above example, the number of iterations $n = 25$.

3. A generalized Lambert function

In this section we consider a formal definition and some analytical properties of the generalized Lambert function. Consider the function $x_\beta : (0, 1) \to \mathbb{R}^1_+$ defined by the formula

$$x_\beta(y) = \begin{cases} -\frac{\log(1-y^\beta)}{y}, & 0 < y < 1, \\ 0, & y = 0, \end{cases} \tag{3.1}$$

where parameter $\beta > 1$ is fixed.

Definition 3.1. The inverse function to x_β is called the *generalized Lambert function*.

We shall denote the generalized Lambert function by $y_\beta(x)$ in what follows.

Since $x_\beta(y)$ is analytic for $y \neq 0, 1$ and continuous at $y = 0$, one can see that $y_\beta(x)$ is a real-valued, analytic function for $x > 0$. Both functions $x_\beta(y)$ and $y_\beta(x)$ are monotonically increasing. Indeed, the derivative of $x_\beta(y)$ is equal to

$$\frac{dx_\beta(y)}{dy} = \frac{\frac{\beta \cdot y^\beta}{1-y^\beta} + \log(1-y^\beta)}{1-y^\beta} > 0 \quad \text{for} \quad y \in (0,1).$$

Then we obtain that

$$\frac{dy_\beta(x)}{dx} > 0 \quad \text{for} \quad x > 0.$$

Therefore, the generalized Lambert function is defined uniquely for any $x > 0$.

Remark 3.2. From (3.1) it immediately follows that the function $x_\beta(y)$ is represented by the absolutely convergent series

$$x_\beta(y) = \sum_{k=1}^{\infty} \frac{y^{\beta k-1}}{k}, \quad 0 \leqslant y < 1. \tag{3.2}$$

Since $\beta > 1$, the derivative

$$\frac{dx_\beta(y)}{dy} = \sum_{k=1}^{\infty} \frac{y^{\beta k-2}}{k(\beta k-1)} > 0$$

for $y \in (0,1)$. Note that $\frac{dx_\beta(y)}{dy}$ blows up at 0 for $1 < \beta < 2$. For $\beta \geqslant 2$ it is defined at 0 by continuity.

Let $\alpha := 1/\beta$. The generalized Lambert function $y_\beta(x)$ obviously satisfies the following transcendental equation:

$$y = \left(1 - e^{-xy}\right)^\alpha, \quad x > 0, \tag{3.3}$$

with $y \in (0,1)$ and parameter $\alpha \in (0,1)$.

Remark 3.3. Equation (2.4) is equivalent to Equation (3.3) in the space of continuous functions, $\mathbf{C}(\mathbb{R}^1_+)$. The role of the variable x in (3.3) plays parameter λ, the variable x in (2.4) is represented by y in (3.3). Equation (3.3) is equivalent to

$$y^\beta = 1 - e^{-xy}, \quad x > 0, \tag{3.4}$$

Remark 3.4. Equation (3.3) has infinitely many discontinuous solutions. Indeed, let $y_\beta^{(c)}$ be a continuous solution of (3.4). Then the function

$$y^*_\beta(y) = \begin{cases} y_\beta^{(c)}(x), & x \neq x_*, \\ 0, & x = x_* \end{cases}$$

is also a solution to (3.3) having discontinuity at x_*.

Now, consider x as a fixed parameter and solve Equation (3.3) with respect to y.

Lemma 3.5. *For each* $x > 0$ *Equation* (3.3) *has two non-negative solutions, one of which is always 0, and the other one belongs to* $(0,1)$.

Proof. The statement of Lemma 3.5 follows from Proposition 2.1. □

The function $\mathsf{y}_\beta(\mathsf{x})$ is intimately connected to the mapping, $\widehat{\mathcal{W}} : [0,1] \to [0,1]$, defined by the relation

$$\widehat{\mathcal{W}}\mathsf{y} = \mathsf{f}(\mathsf{y}),$$

where

$$\mathsf{f}(\mathsf{y}) = \left(1 - e^{-\mathsf{x}\mathsf{y}}\right)^{\alpha}, \quad \mathsf{x} > 0, \tag{3.5}$$

and $\alpha = 1/\beta$. The mapping $\widehat{\mathcal{W}}$ has a trivial fixed point:

$$\mathsf{y}_0 = 0, \qquad \forall \mathsf{x} > 0.$$

Given $\mathsf{x} > 0$, the non-trivial fixed point of $\widehat{\mathcal{W}}$ represents the value of the function, $\mathsf{y}_\beta(\mathsf{x})$, that is a solution of Equation (3.3).

The next statement describes the set of functions, $\mathsf{Y}_\beta = \{\mathsf{y}_\beta(\mathsf{x})\}_{\beta>1}, \mathsf{x} \in \mathbb{R}^1_+$ and delineates the probabilistic nature of the set Y_β.

Let $\mathcal{D}_+$ be the totality of all cumulative distribution functions of non-negative and absolutely continuous random variables.

Proposition 3.6. *For all $\beta > 1$ we have*

$$\mathsf{Y}_\beta \subset \mathcal{D}_+. \tag{3.6}$$

Proof. We need to prove that

i. The limits

$$\lim_{\mathsf{x}\to 0} \mathsf{y}_\beta(\mathsf{x}) = 0, \tag{3.7}$$

and

$$\lim_{\mathsf{x}\to \infty} \mathsf{y}_\beta(\mathsf{x}) = 1. \tag{3.8}$$

ii. The function $\mathsf{y}_\beta(\mathsf{x})$ is monotone.

iii. The function $\mathsf{y}_\beta(\mathsf{x})$ is absolutely continuous.

Proof of i. From the definition of the inverse function, $\mathsf{x}_\beta(\mathsf{y})$, we derive that the limits of the inverse function satisfy the relations

$$\lim_{\mathsf{y}\to 0} \mathsf{x}(\mathsf{y}) = 0 \quad \text{and} \quad \lim_{\mathsf{y}\to 1} \mathsf{x}(\mathsf{y}) = +\infty. \tag{3.9}$$

Then from (3.9) we obtain (3.7) and (3.8).

Proof of ii. Monotonicity of $\mathsf{y}_\beta(\mathsf{x})$ immediately follows from the inequality

$$\frac{\mathrm{d}\mathsf{y}_\beta(\mathsf{x})}{\mathrm{d}\mathsf{x}} > 0 \quad \text{for } \mathsf{x} > 0.$$

Proof of iii. The last statement is obvious.

Thus, the set of functions, Y_β, is a subset of $\mathcal{D}_+$. □

Let us now consider a useful computational scheme for the functions $\mathsf{y}_\beta(\mathsf{x})$. The inverse function, $\mathsf{x}_\beta(\mathsf{y})$, can be naturally used for the computation of $\mathsf{y}_\beta(\mathsf{x})$ for different values of the parameter β, see Figure 2. More precisely, we take a finite set

$$\mathsf{y}_\beta^{(\mathsf{k})} = \frac{\mathsf{k}}{\mathsf{m}}, \quad \mathsf{k} = 1, 2, 3, \ldots, \mathsf{m} - 1,$$

for sufficiently large m and compute the corresponding values of x,

$$x_k = -\frac{\log\left(1-(y_\beta^{(k)})^\beta\right)}{y_\beta^{(k)}}.$$

Hereafter, we shall call the linear interpolation between the nodes $\left\{\left(x_k, y_\beta^{(k)}\right)\right\}_{k=1}^{m-1}$, the *benchmark approximation* of the function $y_\beta(x)$.

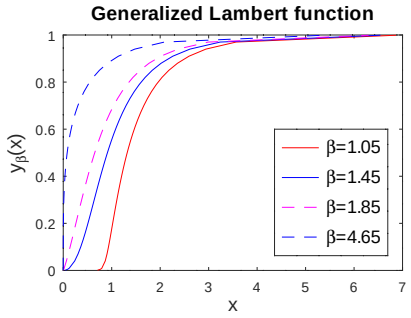

FIGURE 2. Graph of the generalized Lambert function.

Theorem 3.7. *The function $y_\beta(x)$ admits the following continued exponential representation:*

$$y_\beta(x) = \ldots\left(1-\exp\left(-x\cdot\left(1-\exp\left(-x\cdot(1-e^{-x})^\alpha\right)\right)^\alpha\right)\right)^\alpha\ldots \tag{3.10}$$

where $\alpha=\beta^{-1}$. Formula (3.10) should be understood as the limit of the iteration process

$$y_\beta(x) := \lim_{n\to\infty} y_n,$$

where

$$\begin{cases} y_{n+1} &= (1-e^{-x\cdot y_n})^\alpha, \quad n=1,2,\ldots,\\ y_1 &= 1. \end{cases}$$

Given $x>0$, the fixed point of the mapping (3.5) is a non-trivial solution of the transcendental equation (3.3).

Remark 3.8. It is obvious that $\forall n=1,2,\ldots,$ we have that $0<y_n\leqslant 1$.

The proof of Theorem 3.7 is based on the following two important Lemmas 3.9 and 3.10.

Lemma 3.9 (Monotonicity of $\widehat{\mathcal{W}}$)**.** *Consider the sequence $\{y_n\}_{n\geqslant 1}$ defined by the relation $y_{n+1}=\widehat{\mathcal{W}}y_n$ with an arbitrary initial value, $0<y_1\leqslant 1$. Then we have*

$$y_1\geqslant y_2 \Longleftrightarrow y_n\geqslant y_{n+1} \quad \forall\, n=2,3,\ldots; \tag{3.11}$$

$$y_1\leqslant y_2 \Longleftrightarrow y_n\leqslant y_{n+1} \quad \forall\, n=2,3,\ldots. \tag{3.12}$$

Moreover, if $0<y_1\leqslant z_1<1$ and the sequence $\{z_n\}_{n\geqslant 1}$ is defined by the same mapping, $\widehat{\mathcal{W}}$, $z_{n+1}=\widehat{\mathcal{W}}z_n$, then

$$y_n\leqslant z_n \quad n=1,2,\ldots.$$

Proof of Lemma. The proof of the lemma follows immediately from the monotonicity of the function

$$f(y) = \left(1 - e^{-xy}\right)^{\alpha}, \quad 0 < \alpha < 1,\ x > 0,$$

that we obtain from the inequality $\frac{df(y)}{dy} > 0$ for $x > 0$ and $y \in (0, 1]$. Suppose that $1 \geqslant y_1 > y_2 > 0$. Let us prove that $y_2 > y_3$. But $y_2^{\beta} = 1 - e^{-xy_1}$ and $y_3^{\beta} = 1 - e^{-xy_2}$. Since $y_1 > y_2$, we obtain that $y_2^{\beta} > y_3^{\beta}$. Since $\beta > 1$, we derive that $y_2 > y_3$.

A similar reasoning allows us to conclude that if $y_n > y_{n+1}$ then $y_{n+1} > y_{n+2}$, and we obtain that the inequality $y_1 > y_2$ implies that $y_n > y_{n+1}$ for all integer $n \geqslant 2$.

The dual case, $0 < y_1 < y_2 \leqslant 1$ is analogous to the previous one: the latter inequality implies that $y_n < y_{n+1}$ for all integer $n \geqslant 2$.

The proof of the last statement of the Lemma is similar to the proof of the first statement. □

Next, denote

$$g(y) \coloneqq \frac{df(y)}{dy}.$$

Lemma 3.10 (Contraction at fixed point)**.**

i. *The fixed point,* $y_*(x)$*, of the mapping* $\widehat{\mathcal{W}}$ *satisfies the relation*

$$0 < g(y_*) \leqslant \alpha < 1 \qquad \forall x > 0. \tag{3.13}$$

ii. *The derivative,* $g(y)$*, diverges at* 0*:*

$$\lim_{y \to 0} g(y) = +\infty.$$

iii. *The function* $g(y)$ *is monotonically decreasing.*

iv. *For* $y > y_*$*,* $g(y) < \alpha$*.*

Proof of Lemma 3.10. The function, $g(y)$, satisfies the relation

$$g(y) = \alpha \cdot x \cdot \left(1 - e^{-yx}\right)^{\alpha - 1} e^{-yx},$$

and

$$g'(y) = -\alpha x^2 \cdot e^{-yx} \cdot \left(1 - e^{-x \cdot y_*}\right)^{\alpha - 1} - \alpha(1 - \alpha) \cdot e^{-yx} \cdot \left(1 - e^{-x \cdot y_*}\right)^{\alpha - 1} < 0.$$

The latter inequality implies statement (iii).

Denote by $y_*(x)$ the fixed point of the mapping $\widehat{\mathcal{W}}$. Then we have

$$y_* = \left(1 - e^{-x \cdot y_*}\right)^{\alpha}, \quad x > 0.$$

Therefore,

$$\begin{aligned} g(y_*) &= \alpha \cdot x \cdot \left(1 - e^{-x \cdot y_*}\right)^{\alpha - 1} \cdot e^{-x \cdot y_*} \\ &= \alpha \cdot x \cdot \left(1 - e^{-x \cdot y_*}\right)^{\alpha} \cdot \frac{e^{-x \cdot y_*}}{1 - e^{-x \cdot y_*}} \\ &= \alpha \cdot x \cdot y_* \cdot \frac{e^{-x \cdot y_*}}{1 - e^{-x \cdot y_*}} \end{aligned}$$

Denote $t := x \cdot y_*$. Obviously, $t > 0$. Notice that for $t > 0$,

$$e^t - 1 > t.$$

Then we obtain that $g(y_*)$ satisfies the inequality

$$g(y_*) = \alpha \cdot \frac{t}{e^t - 1} \leqslant \alpha < 1.$$

Statement (i) is proved. Therefore, the function f is a contraction near the fixed point y_*, as was to be proved. The proof of statements (ii) and (iv) is straightforward. □

Proof of Theorem 3.7. Fix $x > 0$ and choose $y_1(x) = 1 - e^{-x}$. It is not difficult to see that the sequence $\{y_n\}_{n\geqslant 1}$ satisfies the inequality $0 \leqslant y_n \leqslant 1$ for $n \geqslant 2$ if $y_1 \in (0, 1]$. Monotonicity and boundedness of the sequence $\{y_n\}_{n\geqslant 1}$ implies that there exists the limit

$$y_*(x) = \lim_{n\to\infty} y_n(x).$$

This limit does not depend on the choice of y_1 because convergence to $y_*(x)$ is monotone $\forall x \geqslant 0$, and near the fixed point the mapping is a contraction. The limit, $y_*(x)$, satisfies (3.10) and we obtain that $y_*(x) = y_\beta(x)$ for all $x > 0$. Theorem 3.7 is thus proved. □

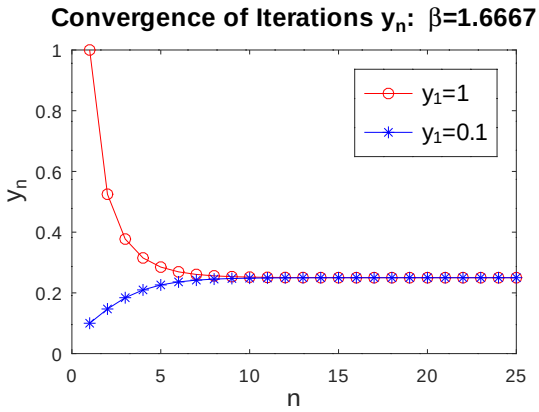

FIGURE 3. Iterations of the mapping $\widehat{\mathcal{W}}$.

Convergence of iterations, $y_n(x)$, is illustrated in Figure 3. The integer parameter, n, is the iteration index, the argument $x = 1.287$ in Figure 3.

The values of the function, $y_\beta(x)$, also depend on the parameter, $\beta > 1$. If parameter β is not fixed, we have a function of two arguments, $y_\beta(x)$.

Proposition 3.11. *The function* $y_\beta(x)$ *is a monotone function of* $\beta, (\beta > 1)$.

Proof. Consider the inverse function $x_\beta(y)$ corresponding to the function $y_\beta(x)$. From the definition of the inverse function (3.1) it follows that $x_\beta(y)$ is monotonically decreasing function of the parameter β. Then we immediately obtain that

the inverse function to $x_\beta(y)$ is monotonically increasing function of β. Proposition 3.11 is thus proved. □

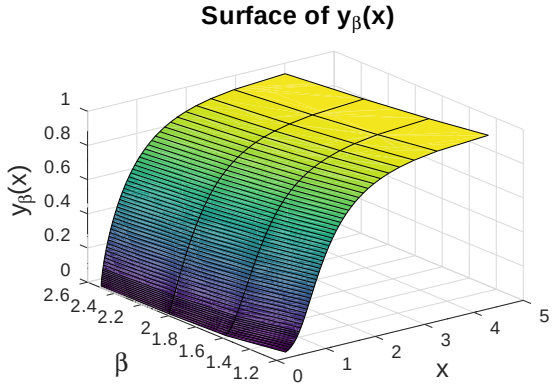

FIGURE 4. Surface of the function $y_\beta(x)$.

Since the limit of iterations, $\{y_n\}_{n\geqslant 1}$, does not depend on the initial value, y_1, we obtain the following

Corollary 3.12. *For any $\theta \in (0,1]$ the function $y_\beta(x)$ admits the following continued exponential representation:*

$$y_\beta(x) = \dots \left(1 - \exp\left(-x \cdot \left(1 - \exp\left(-x \cdot (1 - e^{-x\theta})^\alpha\right)\right)^\alpha\right)\right)^\alpha \dots \tag{3.14}$$

where $\alpha = \beta^{-1} \in (0,1)$.

4. Asymptotic behaviour of function $y_\beta(x)$.

In this section, we consider the asymptotic behaviour of $y_\beta(x)$ for both small and large values of argument x. First, we construct the following asymptotic expansion for function $y_\beta(x)$ in the right-hand neighbourhood of 0.

Proposition 4.1. *The function, $y_\beta(x)$, has the following asymptotic on the lower tail:*

$$y_\beta(x) = \frac{x^{\frac{1}{\beta-1}}}{1 + \frac{x^{\frac{\beta+1}{\beta-1}}}{2(\beta-1)}} + o\left(x^{\frac{\beta+2}{\beta-1}}\right) \quad \text{as } x \downarrow 0. \tag{4.1}$$

Proof. We have

$$1 - e^{-xy_\beta} = xy_\beta - \frac{1}{2}x^2y_\beta^2 + o(x^2).$$

Let us rewrite Equation (3.3) as

$$y^\beta = 1 - e^{-xy}, \quad x > 0,$$

Then we obtain from the latter equation that

$$(y_\beta(x))^{\beta-1} = x \cdot \left(1 - \frac{1}{2}xy_\beta(x) + o(x)\right). \tag{4.2}$$

Let $\gamma := \beta - 1$. Since $\beta > 1$, the parameter γ is positive and hence, we get for small t that $(1-t)^{\gamma} = 1 - \gamma t + o(t)$.

From (4.2) and the latter asymptotic formula we obtain that $y_\beta(x)$ satisfies the following relation:

$$y_\beta(x) = x^{\frac{1}{\beta-1}} \left(1 - \frac{1}{2(\beta-1)} x y_\beta(x) + o\left(x^{\frac{1}{\beta-1}}\right)\right).$$

Then we derive that

$$y_\beta(x) \cdot \left(1 + \frac{x^{\frac{\beta}{\beta-1}}}{2(\beta-1)}\right) = x^{\frac{1}{\beta-1}} + o\left(x^{\frac{2}{\beta-1}}\right),$$

and, finally,

$$y_\beta(x) = \frac{x^{\frac{1}{\beta-1}}}{1 + \frac{x^{\frac{\beta}{\beta-1}}}{2(\beta-1)}} + o\left(x^{\frac{2}{\beta-1}}\right).$$

The proposition is thus proved. □

The asymptotics (4.1) of $y_\beta(x)$, as $x \to 0$, is just a starting point. It can be improved as follows:

Proposition 4.2. *The function,* $y_\beta(x)$ *admits the following asymptotic expansion as* $x \downarrow 0$*:*

$$y_\beta(x) = x^{\frac{1}{\beta-1}} \cdot \left(1 - \frac{1}{2(\beta-1)} x^{\frac{\beta}{\beta-1}} + \frac{\beta+8}{24(\beta-1)^2} x^{\frac{2\beta}{\beta-1}} - \frac{\beta+3}{12(\beta-1)^3} x^{\frac{3\beta}{\beta-1}} + \dots\right) \quad (4.3)$$

The proof of this proposition is deferred to the Appendix.

Corollary 4.3. *The function* $\hat{z}(x) := y_\beta(x) \cdot x^{\frac{1}{1-\beta}}$ *satisfies the following limit relation*

$$\lim_{x \downarrow 0} \hat{z}(x) = 1.$$

Remark 4.4. Let us introduce a new variable, $t := x^{\frac{\beta}{\beta-1}}$ and define

$$z(t) := \hat{z}(x).$$

The function $z(t)$ satisfies the following transcendental equation:

$$z^{\beta-1}(t) = \varphi\left(t \cdot z(t)\right),$$

where

$$\varphi(u) = \frac{1 - e^{-u}}{u}.$$

It is instructive to look at the graph of the function $z(t)$. In Figure 5 the graph of the function $z(t)$ is shown for the following three particular values of β: $\beta_1 = 1.1$, $\beta_2 = 1.15$, $\beta_3 = 1.3$. All three graphs, obviously, start at the point $(0, 1)$. The function $z(t)$ is downward convex, positive and monotonically decreasing.

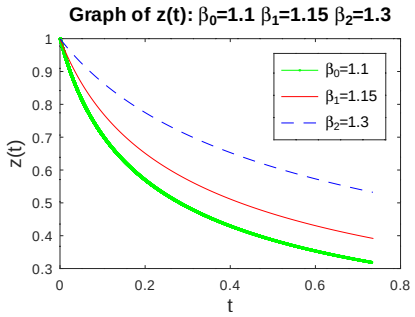

FIGURE 5. Graph of the function $z(t)$.

Conjecture. *The function* $z(t)$, $(t \geqslant 0)$, *is analytical.*

Our next step is the derivation of the two-sided bounds for the function $y_\beta(x)$. Numerical comparison of the approximations (4.1) and (4.3) and the lower bound, $y_L(x)$, introduced in Proposition 4.5, is discussed in Section 5.

Proposition 4.5. *For* $x > 0$ *and* $\beta > 1$, *the function* $y_\beta(x)$ *satisfies the following two-sided inequality:*

$$\left(1 - e^{-x}\right)^{\frac{1}{\beta-1}} < y_\beta(x) < \left(1 - e^{-x}\right)^{\frac{1}{\beta}} \tag{4.4}$$

Proof. It follows from the definition of the inverse function that

$$x = y_\beta^{\beta-1} + \frac{1}{2} y_\beta^{2\beta-1} + \frac{1}{3} y_\beta^{3\beta-1} + \dots$$

Since $0 < y_* < 1$, we have

$$x < y_\beta^{\beta-1} + \frac{1}{2} y_\beta^{2\beta-2} + \frac{1}{3} y_\beta^{3\beta-3} + \dots$$

The latter inequality can be written as

$$x < -\log\left(1 - y_\beta^{\beta-1}\right).$$

Therefore,

$$y_\beta(x) > \left(1 - e^{-x}\right)^{\frac{1}{\beta-1}}.$$

On the other hand, since

$$x = -\frac{\log(1 - y_\beta^\beta)}{y_*} \quad \text{and} \quad 0 < y_\beta < 1,$$

we obtain that

$$x > -\log(1 - y_\beta^\beta)$$

The latter inequality is equivalent to $y_\beta(x) < (1 - e^{-x})^{\frac{1}{\beta}}$. The upper bound, $y^U(x) = (1 - e^{-x})^\alpha$, and the lower bound, $y_L(x) = (1 - e^{-x})^{\frac{\alpha}{1-\alpha}}$, hence established. □

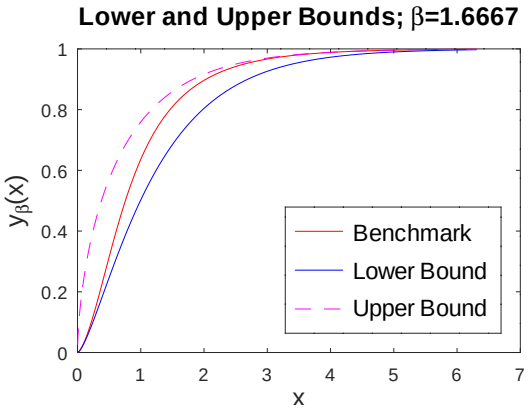

FIGURE 6. The function $y_\beta(x)$ and the bounds (4.4).

In Figure 6 the graph of the function, $y_\beta(x)$, is displayed together with the upper and lower bounds established in Proposition 4.5. The lower bound captures the asymptotic behaviour of the function y_β as $x \downarrow 0$ while the upper bound captures the asymptotic behaviour of $y_\beta(x)$ as $x \to \infty$.

It is also not difficult to see that the ratio of the bounds approaches 1 at infinity

$$\lim_{x\to\infty} \frac{y^U(x)}{y_L(x)} = 1. \tag{4.5}$$

Indeed, the ratio $\frac{y^U(x)}{y_L(x)}$, can be written as

$$\frac{y^U(x)}{y_L(x)} = \left(1 + \frac{1}{e^x - 1}\right)^\kappa,$$

where $\kappa := \frac{\alpha^2}{1-\alpha} > 0$. The latter relation implies the validity of (4.5).

5. Numerical examples

5.1. Roots of Equation (2.4). The problem of the computation of the extinction probability can be addressed by using several numerical methods.

The first approach is related to solution of Equation (2.4), considered in Section 2, by bisection method. An important technical issue is due to the fact for the values of parameter β close to 1 this method often presents a difficulty in finding the interval containing the root of the equation. This situation is illustrated in Figure 7 where the graph of the function, $\mathcal{F}_\beta(x)$, satisfying Equation (2.5), is shown for three different values of parameter β: $\beta_1 = 1.25$, $\beta_2 = 1.6$ and $\beta_3 = 1.95$.

The positive root of the equation, marked by the star in Figure 7 is moving towards 0 as $\beta \downarrow 1$.

Moreover, the minimal value of the function tends to 0 monotonically, namely, $\inf_{x>0} \mathcal{F}_\beta(x) \downarrow 0$ as $\beta \downarrow 1$. For this reason, the root of the equation, $x_0(\beta) \downarrow 0$, as $\beta \downarrow 1$. Such a behaviour represents certain computational difficulties for the bisection method. The numerical experiments with MatLab demonstrate that the

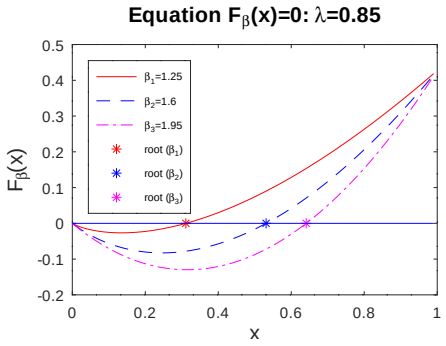

FIGURE 7. Roots of Equation $\mathcal{F}_\beta(x) = 0$.

interval, $[a, b]$, such that $\mathcal{F}_\beta(a) \cdot \mathcal{F}_\beta(b) < 0$, remains undetermined for $\beta < 1.1$, due to computer finite precision.

5.2. Comparison of approximations. A much more efficient solution can be obtained by using the asymptotic approximations, derived in Section 4. Recall that the lower bound used as the asymptotic approximation of the function, $y_\beta(x)$, is $y_L(x) = (1 - e^{-x})^{\frac{\alpha}{1-\alpha}}$.

The first and the second approximations, $y_\beta^{(1)}(x)$ and $y_\beta^{(2)}(x)$ of the function $y_\beta(x)$, defined by Equations (4.1) and (4.3), respectively, and the lower bound, $y_L(x)$, are compared with the benchmark approximation of $y_\beta(x)$. The methodology of comparison is founded on a fixed set, $Y = \{y_1, y_2, \dots, y_n\}$, of values of function, where y_j are close to 0. Given $\beta > 1$, we compute the benchmark approximation, $x_j = x_\beta(y_j)$, $(j = 1, 2, \dots, n)$, and the distances $d_i(x) = |\, y_\beta^{(i)}(x) - y_\beta(x)\,|$, $(i = 1, 2)$, and $d_L(x) = |\, y_L(x) - y_\beta(x)\,|$ for each $x = x_j$.
In Figure 8 we compare logarithms of the distances, $\log(d_i(x))$, $(i = 1, 2)$ and $\log(d_L(x))$. Parameter $\beta = 1.25$. The set Y is an arithmetic progression,

$$Y = \{y_k\}_{k \geqslant 1}, \quad y_k = y_1 + \Delta \cdot (k - 1),$$

where $y_1 = 0.0001$, $\max\limits_{k \geqslant 1} y_k = 0.15$, $\Delta = 0.0001$. The results shown in Figure 8 demonstrate advantage of the approximation $y_\beta^{(2)}(x)$.

Similar results are obtained for the other values of parameter β, see Figure 9. In that Figure, parameter β takes two values, $\beta_1 = 1.25$ and $\beta_2 = 1.35$.

In both cases the numerical results demonstrate advantage of approximation (4.3).

5.3. The number of iterations required. The iterations described in Theorem 3.7 converge independently of the choice of the initial approximation, y_1. However, the number of iterations, $\mathcal{N}(\varepsilon)$, required for a specified accuracy $\varepsilon > 0$, does depend on y_1 as well as on β and x. Let us define $\mathcal{N}$ more precisely.

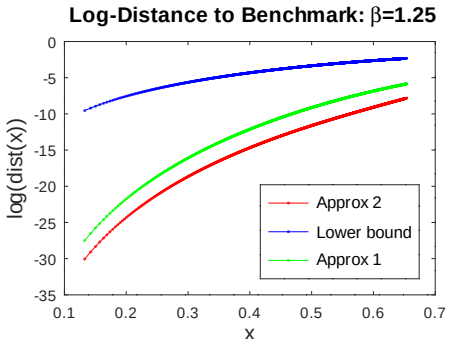

FIGURE 8. Log-distance to benchmark.

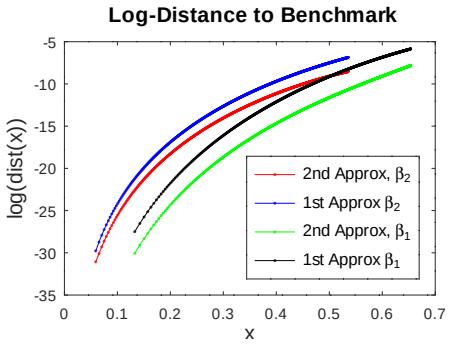

FIGURE 9. Log-distance to benchmark: $\beta_1 = 1.25, \beta_2 = 1.35$.

Definition 5.1. Given $x > 0$, the required number of iterations, $\mathcal{N}(\varepsilon, x)$, is defined as follows:

$$\mathcal{N}(\varepsilon, x) := \min_{n \geqslant 1} |\, y_n - y_\beta(x)\,| \leqslant \varepsilon.$$

Let us compare the following strategies for the choice of y_1:

1. $y_1^{(1)}(x) = (1 - e^{-x})^{\frac{1}{\beta}}$ for all $x > 0$.
2. $y_1^{(2)}(x) = (1 - e^{-x})^{\frac{1}{\beta - 1}}$.
3. $y_1^{(3)}(x) = \frac{1}{2} \cdot \left((1 - e^{-x})^{\frac{1}{\beta - 1}} + (1 - e^{-x})^{\frac{1}{\beta}} \right)$.

The first strategy always takes the initial approximation to coincide with the upper bound. The second strategy takes the initial approximation to be equal to the lower bound for the solution. The third strategy takes the mid-point as the initial approximation.

In Figure 10, the above three strategies are compared for $\varepsilon = 10^{-5}$. Parameter $\beta = 1.25$ in the numerical experiment related to Figure 10 is described in this section.

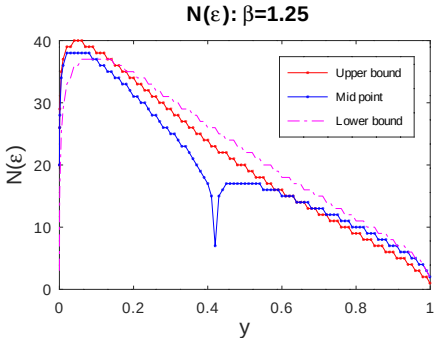

FIGURE 10. Required number of iterations, $\mathcal{N}(\varepsilon)$.

It is convenient to change the argument x in the definition of the required number of iterations to $y_\beta(x)$. In this case, the second argument of the function $\mathcal{N}(\varepsilon, y)$ will belong to the interval $[0, 1]$.

While the argument $y = y_\beta(x)$ is small enough, $0 < y \leqslant 0.15$, the initial approximation, $y_1^{(2)}(x)$, dominates the other strategies. In the mid-range case, $0.15 < y \leqslant 0.6$, the mid-point strategy, $y_1^{(3)}(x)$, provides a faster convergence than the remaining two strategies.

For the large values of the argument x, such that $y \geqslant 0.6$, where $y = y_\beta(x)$, the first strategy is the most efficient. This fact is consistent with the property of the upper bound, $y^U(x)$, to provide the asymptotic approximation of $y_\beta(x)$ as $x \to \infty$.

6. Definition and properties of the random variable ξ_β

In the previous section we demonstrated that for each $\beta > 1$ the function $y_\beta(x)$ is the cdf of an absolutely continuous random variable denoted by ξ_β. Despite the fact that the cdf of ξ_β is not known explicitly, many characteristics of ξ_β can still be found in closed form. In the next subsection, we compute the moments of this random variable.

6.1. Computation of the moments, $\mathbb{E}[\xi_\beta^n]$.

Definition 6.1. We say that the random variable ξ_β has the generalized Lambert distribution if

$$\mathbb{P}(\xi_\beta \leqslant x) = y_\beta(x), \quad \forall x \geqslant 0.$$

Proposition 6.2. *The nth moment of the r.v. ξ_β is*

$$\mathbb{E}[\xi_\beta^n] = (-1)^n \alpha \cdot \int_0^1 \frac{\log^n(1-t)}{t^{(n-1)\alpha+1}}\, dt, \tag{6.1}$$

where $\alpha = 1/\beta$.

Proof. Let ξ_β be defined on a certain probability space $(\Omega, \mathfrak{B}, \mathbb{P})$ and let $\mathcal{U} = \mathcal{U}([0,1])$ be a random variable on the same probability space, $(\Omega, \mathfrak{B}, \mathbb{P})$, having uniform $[0,1]$ distribution. The random variable ξ_β can be written as follows:

$$\xi_\beta = -\frac{\log\left(1-\mathcal{U}^\beta\right)}{\mathcal{U}}. \tag{6.2}$$

Then we obtain from (6.2) that the moments of the r.v. ξ_β satisfy the relation

$$\mathbb{E}[\xi_\beta^n] = \int_0^1 (-1)^n s^{-n} \cdot \log^n\left(1-s^\beta\right)\, ds, \quad n = 1, 2, \ldots.$$

Changing the variables $t := s^\beta$, in the above integral ascertains the validity of (6.1). □

In Figure 11 the first 6 log-moments, $\log\left(\mathbb{E}[\xi_\beta^n]\right)$ of the random variable ξ_β are shown for $\beta = 1.2$. The theoretical results are obtained from (6.1). These results are compared with that obtained by Monte Carlo method.

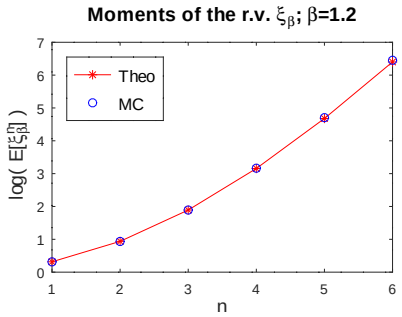

FIGURE 11. The log-moments of ξ_β: $\log\left(\mathbb{E}[\xi_\beta^n]\right), n = 1, 2, \ldots, 6$.

The following statement stipulates integrability of the function $1 - y_\beta(x)$ and demonstrates unexpected connection to the Riemann zeta function, $\zeta(x)$.

Proposition 6.3. *The function $y_\beta(x)$ satisfies the following relation:*

$$\int_0^\infty (1 - y_\beta(x))\, dx = \frac{\pi^2}{6} \cdot \frac{1}{\beta}. \tag{6.3}$$

Proof. Since the random variable $\xi_\beta \geqslant 0$, its first moment can be obtained by integrating the corresponding survival function:

$$\mathbb{E}[\xi_\beta] = \int_0^\infty (1 - y_\beta(x))\, dx.$$

On the other hand, from (6.1) we have

$$\mathbb{E}[\xi_\beta] = -\int_0^1 \frac{\log\left(1-s^\beta\right)}{s}\, ds. \tag{6.4}$$

Denote $\nu := s^\beta$ and note that for $0 \leqslant s < 1$,

$$\log(1-\nu) = -\sum_{n=1}^{\infty} \frac{\nu^n}{n}.$$

Then from (6.4) we derive

$$\begin{aligned} \mathbb{E}[\xi_\beta] &= \int_0^1 \sum_{n=1}^{\infty} \frac{s^{\beta n-1}}{n}\, ds \\ &= \frac{1}{\beta} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

Thus, formula (6.3) is derived. □

Obviously, we can write the first moment of ξ_β as $\mathbb{E}[\xi_\beta] = \zeta(2) \cdot \alpha$.

Proposition 6.4. *The second moment of the random variable, ξ_β, satisfies the relation*

$$\mathbb{E}[\xi_\beta^2] = \frac{1}{\beta} \cdot \int_0^1 t^{-1-\alpha} \log^2(1-t)\, dt. \tag{6.5}$$

The standard deviation of ξ_β is

$$\sigma(\xi_\beta) = \sqrt{\alpha \cdot \int_0^1 t^{-1-\alpha} \log^2(1-t)\, dt - \frac{\pi^4}{36} \cdot \alpha^2}. \tag{6.6}$$

Proof. The proof of (6.5) follows from Equation (6.1). The formula for the standard deviation, $\sigma(\xi_\beta)$, follows from (6.5) and the relation $\mathbb{E}[\xi_\beta] = \frac{\pi^2}{6} \cdot \alpha$. □

In Figure 12, the first moment, $E(\alpha) = \mathbb{E}[\xi_{1/\alpha}]$ and the standard deviation, $\sigma(\alpha) = \sigma(\xi_{1/\alpha})$ are shown for $\alpha \in (0,1)$. The analytical results are accompanied by the Monte Carlo valuation of the first moment of ξ_β and the standard deviation of this random variable.

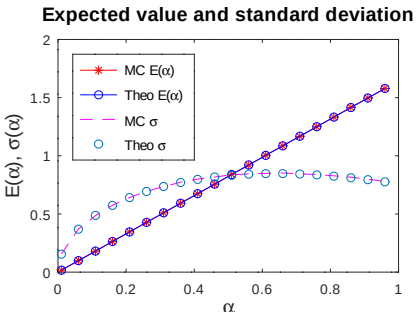

FIGURE 12. Expectation and standard deviation of ξ_β.

The theoretical value of the expectation, $\mathbb{E}[\xi_\beta]$, and its MC estimator form a straight line in Figure 12. The standard deviation, $\sigma(\alpha)$, and its MC estimator exhibit a non-linear, non-monotone dependency on α.

6.2. The moment problem for ξ_β. The classical moment problem [14], [21], deals with existence and uniqueness of a finite, non-negative measure μ on $\mathbb{R}^1$ such that a given sequence of real numbers, $m_1, m_2, \ldots, m_n, \ldots$ satisfying the relation

$$m_n = \int_{\mathbb{R}^1} x^n \, d\mu(x), \quad n = 0, 1, 2, \ldots.$$

In this Section, we consider the case of the probability measure μ corresponding to the distribution of the random variable ξ_β:

$$\mu(A) = \mathbb{P}\left(\xi_\beta \in A\right), \quad A \text{ is a measurable subset of } \mathbb{R}^1.$$

In this case, the support of the measure μ is $\mathbb{R}^1_+$.

Existence of such probability measure is obvious in our case. The uniqueness problem can be formulated as follows:

> *Given a sequence of moments, $\{M_n\}_{n\geqslant 1}$, is there another measure, μ^*, different from the probability measure μ defined by the cumulative distribution function $y_\beta(x) = \mathbb{P}(\xi_\beta \leqslant x)$, such that the nth moment corresponding to measure μ^* is M_n for all integer $n \geqslant 1$?*

If the solution to the moment problem is unique, we say that the moment problem is determinate. Otherwise the moment problem is said to be indeterminate.

Proposition 6.5. *Let the moments, $\{M_n\}_{n\geqslant 1}$, be defined by Equation* (6.1)*:*

$$M_n = (-1)^n \alpha \cdot \int_0^1 \frac{\log^n(1-s)}{s^{(n-1)\alpha+1}} \, ds.$$

Then there exists a unique probability measure, μ, on $\mathbb{R}^1_+$ such that

$$\int_0^\infty x^n \, d\mu(x) = M_n, \quad n = 0, 1, \ldots.$$

The proof of Proposition 6.5 is based on the following technical statement on the rate of growth of the moments, M_n.

Lemma 6.6. *The sequence M_n exhibits the following asymptotic relation:*

$$M_n = c_\beta \cdot n! + \mathcal{O}\left(\frac{n!}{2^n}\right) \quad \text{as } n \to \infty, \tag{6.7}$$

where the constant c_β satisfies the inequalities $\frac{1}{\beta} < c_\beta < \frac{1}{\beta-1}$.

Proof of Lemma 6.6 is deferred to the Appendix.

Proof of Proposition 6.5. Let us prove that the measure μ is determinate. We would like to verify the following sufficient condition for the moments of the probability measure concentrated on $\mathbb{R}^1_+$ (see [14], [21], [24]):

$$\limsup_{n\to\infty} \frac{M_n^{1/n}}{n} < \infty \tag{6.8}$$

From (6.7) we derive that as $n \to \infty$

$$M_n = c_\beta \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n + o(n!).$$

Then we find that

$$\limsup_{n\to\infty} \frac{M_n^{1/n}}{n} = \mathcal{O}(1).$$

The latter asymptotic equation implies (6.8), as was to be proved. □

Remark 6.7. One can also use Carleman's test (see [14], [21], [24]) to verify that the sequence of moments, $\{M_n\}_{n\geqslant 1}$ is determinate.

6.3. Moments, generating function and identities. We conclude this section with a few remarks on the connection between the moments of the r.v. ξ_β and the generating functions of several remarkable sequences including the sequence $\{\zeta(n)\}_{n\geqslant 2}$, where $\zeta(\cdot)$ is the Riemann zeta function.

Let us start with the second moment, $\mathbb{E}[\xi_\beta^2]$. From (6.5) we have

$$\mathbb{E}[\xi_{1/\alpha}^2] = \alpha \cdot \mathcal{I}(\alpha), \tag{6.9}$$

where

$$\mathcal{I}(\alpha) = \int_0^1 t^{-1-\alpha} \log^2(1-t)\, dt.$$

Proposition 6.8. *The integral,* $\mathcal{I}(\alpha)$*, satisfies the relation*

$$\mathcal{I}(\alpha) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{k}\cdot\frac{1}{m}\cdot\frac{1}{k+m-\alpha}, \qquad 0<\alpha<1. \tag{6.10}$$

Proposition 6.8 is proved in the Appendix.

Let us now represent the integral, $\mathcal{I}(\alpha)$, as the generating function of a harmonic sequence. Denote by H_n, the sum of the harmonic series

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad n = 1, 2, \ldots,$$

and for $n = 0$ we define $H_0 = 0$. Obviously, we have

$$H_n = H_{n-1} + \frac{1}{n}, \quad n = 1, 2, \ldots.$$

Let us now define the sequence

$$A_\ell := 2 \cdot \sum_{n=1}^{\infty} \frac{H_{n-1}}{n^{\ell+2}}, \quad \ell = 0, 1, 2, \ldots. \tag{6.11}$$

Proposition 6.9. *The elements of the sequence* $\{A_\ell\}_{\ell\geqslant 1}$ *satisfy the relation*

$$A_\ell = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{k}\cdot\frac{1}{m}\cdot\frac{1}{(k+m)^{\ell+1}}, \quad \ell = 0, 1, 2, \ldots. \tag{6.12}$$

The integral $\mathcal{I}(\alpha)$ *is the generating function of the sequence* $\{A_\ell\}_{\ell\geqslant 1}$*:*

$$\mathcal{I}(\alpha) = \sum_{\ell=0}^{\infty} A_\ell \alpha^\ell. \tag{6.13}$$

The proof of Proposition 6.9 is deferred to the Appendix.

The double sums similar to (6.12) are usually called *Tornheim series or Tornheim double sums*. They find applications in many areas of mathematics and physics, see [15], [25].

Remark 6.10. The sequence $\{A_\ell\}_{\ell\geqslant 0}$ can also be expressed in terms of the Riemann zeta function as follows

$$A_\ell = 2\sum_{n=1}^{\infty}\frac{H_n}{n^{\ell+2}} - 2\zeta(\ell+3).$$

The coefficient, A_0, determines the asymptotic behaviour of the second moment as $\alpha \downarrow 0$.

Proposition 6.11.

$$A_0 = 2\zeta(3) = 2.404138\ldots. \tag{6.14}$$

The proof of this proposition is deferred to the Appendix.

Corollary 6.12. *The variance of the r.v.* ξ_β *admits the following asymptotics as* $\alpha \downarrow 0$*:*

$$\sigma^2(\xi_\beta) = A_0\cdot\alpha + o(\alpha).$$

Proof. The statement of the corollary immediately follows from Equations (6.9) and (6.13). □

Computation of the coefficients, A_1, A_2, ..., A_N requires computation of the harmonic sums

$$S_k = \sum_{n=1}^{\infty}\frac{H_n}{n^k}, \quad k = 2, 3, \ldots.$$

Corollary 6.13 (compare with [1], [11])**.**

i. *The coefficient* A_1 *satisfies the relation*

$$A_1 = 2\sum_{k=1}^{\infty}\frac{H_k}{k^3} - 2\zeta(4) = \frac{\pi^4}{180}.$$

ii. *The harmonic sum*

$$\sum_{k=1}^{\infty}\frac{H_k}{k^3} = \frac{\pi^4}{72}.$$

Numerous representations of such character for harmonic sums can also be found in [11].

Analysis of the higher moments of ξ_β leads to relations generalizing properties of harmonic double sums. This line of research will be considered in our subsequent work.

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Appendix A. Proof of Technical Results

Proof of Proposition 4.2. Let us represent the function, y_β, as

$$y_\beta(x) := x^{\frac{1}{\beta-1}} \cdot \hat{z}(x). \tag{A.1}$$

Then, obviously,

$$\hat{z}(x) = y_\beta(x) \cdot x^{\frac{1}{1-\beta}}.$$

From Equation (4.1) it follows that the limit

$$\lim_{x \downarrow 0} \hat{z}(x) = 1.$$

It is convenient to introduce the variable $t := x^{\frac{\beta}{\beta-1}}$. Denote by $z(t) := \hat{z}(x)$. Then we obtain that the function $z(t)$ satisfies the implicit equation

$$z^{\beta-1}(t) = 1 - \frac{1}{2} z(t) \cdot t + \frac{1}{6}(z(t) \cdot t)^2 - \frac{1}{24}(z(t) \cdot t)^3 + \dots \tag{A.2}$$

Let us represent the function $z(t)$ in the following form:

$$z(t) := 1 + \sum_{n=1}^{\infty} z_n t^n. \tag{A.3}$$

Then substituting (A.3) into (A.2), we recursively derive the first coefficients in Expansion (A.3):

$$z_1 = -\frac{1}{2(\beta-1)}, \quad z_2 = \frac{\beta+8}{24(\beta-1)^2}, \quad z_3 = -\frac{\beta+3}{12(\beta-1)^3}.$$

The proposition is thus proved. □

Proof of Lemma 6.6. We have to prove that the moments M_n have the following asymptotics:

$$M_n = c_\beta \cdot n! + o(n!) \quad \text{as } n \to \infty,$$

where the constant c_β satisfies the inequality $\frac{1}{\beta} < c_\beta < \frac{1}{\beta-1}$. Since the support of the measure μ is $\mathbb{R}^1_+$, the n^{th} moment, M_n, can be evaluated as follows:

$$M_n = n \int_0^\infty x^{n-1} \bar{F}(x)\, dx,$$

where the survival function $\bar{F}(x) = \mu([x, +\infty))$. Therefore,

$$M_n = n \int_0^\infty x^{n-1} \bar{y}_\beta(x)\, dx, \tag{A.4}$$

where $\bar{y}_\beta(x) = 1 - y_\beta(x)$. Combining (4.4) and (A.4) we derive the inequalities

$$\int_0^\infty x^{n-1}\Big(1 - (1 - e^{-x})^{1/\beta}\Big)\, dx < \frac{M_n}{n} < \int_0^\infty x^{n-1}\Big(1 - (1 - e^{-x})^{\frac{1}{\beta-1}}\Big)\, dx. \tag{A.5}$$

Next, consider the integral

$$\mathcal{J}(\gamma) := n \int_0^\infty x^{n-1}\left(1 - (1 - e^{-x})^{1/\gamma}\right) dx, \quad \gamma > 0.$$

Taking into account that

$$(1-e^{-x})^{1/\gamma} = 1 - \frac{1}{\gamma}e^{-x} + \frac{1-\gamma}{2\gamma^2}e^{-2x} + o\left(e^{-2x}\right)$$

as $x \to \infty$, we obtain the following asymptotic relation as $n \to \infty$

$$\mathcal{J}(\gamma) = \frac{1}{\gamma}n\Gamma(n) - \frac{n\cdot\Gamma(n)}{2^n}\frac{1-\gamma}{\gamma^2} + o\left(\frac{n!}{2^n}\right), \tag{A.6}$$

where $\Gamma(n)$ is the Gamma function. Then from (A.5) and (A.6) we obtain that

$$\limsup_{n\to\infty}\frac{M_n}{n!} = c_\beta, \quad \frac{1}{\beta} \leqslant c_\beta \leqslant \frac{1}{\beta-1}.$$

The lemma is thus proved. □

Proof of Proposition 6.8. Recall the representation of the function $\log(1-t)$ by the following convergent series:

$$-\log(1-t) = \sum_{k=1}^{\infty}\frac{t^k}{k}, \quad 0 \leqslant t < 1.$$

Hence, the integral $\mathcal{I}(\alpha)$ can be written as follows:

$$\mathcal{I}(\alpha) = \int_0^1 t^{-1-\alpha}\sum_{k=1}^{\infty}\frac{t^k}{k}\cdot\sum_{m=1}^{\infty}\frac{t^m}{m}\,dt.$$

The latter relation implies that

$$\begin{aligned}\mathcal{I}(\alpha) &= \int_0^1 t^{-1-\alpha}\sum_{k=1}^{\infty}\frac{t^k}{k}\cdot\sum_{m=1}^{\infty}\frac{t^m}{m}\,dt\\ &= \sum_{k=1}^{\infty}\frac{1}{k}\cdot\sum_{m=1}^{\infty}\frac{1}{m}\int_0^1 t^{k+m-\alpha-1}\,dt\\ &= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{k\cdot m\cdot(k+m-\alpha)}.\end{aligned}$$

The proposition is thus proved. □

Proof of Proposition 6.9. Consider the double sum in (6.10). We have that for $0 < \alpha < 1$,

$$\frac{1}{k+m-\alpha} = \frac{1}{k+m}\cdot\sum_{\ell=0}^{\infty}\frac{\alpha^\ell}{(k+m)^\ell}.$$

Then we derive that

$$\begin{aligned}\mathcal{I}(\alpha) &= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{k}\cdot\frac{1}{m}\cdot\frac{1}{k+m-\alpha}\\ &= \sum_{\ell=0}^{\infty}\alpha^\ell\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{k}\cdot\frac{1}{m}\cdot\frac{1}{(k+m)^{\ell+1}}.\end{aligned}$$

The latter relation implies (6.12). Now, consider the inner double sum,

$$A_\ell = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{k}\cdot\frac{1}{m}\cdot\frac{1}{(k+m)^{\ell+1}}.$$

Changing the summation index, $n = k + m$, we obtain that

$$\begin{aligned} A_\ell &= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{k}\cdot\frac{1}{m}\cdot\frac{1}{(k+m)^{\ell+1}} \\ &= \sum_{n=2}^{\infty}\frac{1}{n^{\ell+2}}\sum_{k=1}^{n-1}\left(\frac{1}{k}+\frac{1}{n-k}\right). \end{aligned}$$

It is obvious that the latter sum is equal to $2H_{n-1}$. Therefore,

$$\mathcal{I}(\alpha) = 2\sum_{\ell=0}^{\infty}\alpha^\ell\sum_{n=2}^{\infty}\frac{H_{n-1}}{n^{\ell+2}}.$$

From the definition of the sequence A_ℓ (see Equation (6.11)) we obtain that

$$\mathcal{I}(\alpha) = \sum_{\ell=0}^{\infty} A_\ell\alpha^\ell,$$

as was to be proved. □

Proposition 6.11. Let us prove that $A_0 = 2\zeta(3)$. From (6.12) we have

$$A_0 = \sum_{k\geqslant 1}\sum_{m\geqslant 1}\frac{1}{k\cdot m\cdot(k+m)}.$$

Then we derive that

$$\begin{aligned} \sum_{k\geqslant 1}\sum_{m\geqslant 1}\frac{1}{k\cdot m\cdot(k+m)} &= \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{1}{n}\cdot\frac{1}{k\cdot(n-k)} \\ &= \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{1}{n^2}\left(\frac{1}{k}+\frac{1}{(n-k)}\right) \\ &= 2\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\left(H_n-\frac{1}{n}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{k\geqslant 1}\sum_{m\geqslant 1}\frac{1}{k\cdot m\cdot(k+m)} &= \sum_{k\geqslant 1}\sum_{m\geqslant 1}\frac{1}{k^2}\cdot\left(\frac{1}{m}-\frac{1}{k+m}\right) \\ &= \sum_{k\geqslant 1}\frac{1}{k^2}\cdot H_k. \end{aligned}$$

Thus, we obtain that

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\cdot H_k = 2\cdot\sum_{k=1}^{\infty}\frac{1}{k^3} = 2\zeta(3).$$

Formula (6.14) is hence derived. □

Alexander Kreinin: Department of Computer Science, University of Toronto, Toronto, ON, M5S 2E4, CANADA
Email address: alexander.kreinin@utoronto.ca

Andrey Marchenko: LayerOneFinancial
Email address: avmarch@gmail.com

Vladimir V. Vinogradov: Department of Mathematics, Ohio University, Athens, OH 45701 USA
Email address: vinograd@ohio.edu

Vladimir V. Vinogradov: Department of Computer and Mathematical Sciences, IA 4118, University of Toronto Scarborough, 1265 Military Trail, Toronto, Ontario M1C 1A4, Canada
Email address: vv.vinogradov@utoronto.ca